THERMALLY DEVELOPING POISEUILLE FLOW AFFECTED BY RADIATION

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A combined convective and radiative heat transfer problem in thermally developing Poiseuille flow in a cylindrical tube is analyzed. A complex form of the nonlinear integrodifferential radiative transfer equation is solved by the discrete ordinates method in an axisymmetric geometry. To check its accuracy, the solution obtained by the discrete ordinates method is compared with that solved by the integral method. A parametric study is also performed for the conduction-to-radiation parameter, optical thickness, wall emissivity, scattering albedo, and linear anisotropic scattering coefficient. The results show a significant effect of the radiation on the thermal characteristics.

INTRODUCTION

A number of studies for radiation combined with other modes of heat transfer in a multidimensional participating medium have been performed in the past decades. However, since it is nearly impossible to find an exact analytical solution to the highly nonlinear integrodifferential radiative transfer equation, an efficient tool to deal with multidimensional radiative heat transfer is in strong demand to analyze variously coupled thermal problems. The current study discusses thermally developing Poiseuille flow in a cylindrical tube.

While the interaction of one-dimensional (1-D) thermal radiation with forced convection in the thermal entrance region of a circular pipe has been studied by many investigators [1–6], only a few have examined the problem of 2-D thermal radiation combined with forced convection. Chung and Kim [7] and Kassemi and Chung [8] investigated a thermally developing channel flow with 2-D analysis of radiation using the finite element method and zone method, respectively. Echigo et al. [9] studied 2-D development of thermal radiation in an infinite circular tube flow with constant wall temperature by the integral method. Huang and Lin [10] analyzed thermally developing flow in a cylindrical tube with uniform wall heat flux by the direct integral method.

There are various solution methods for solving multidimensional radiative transfer equations. Although the zone [8] method and Monte-Carlo [11] technique
have long been accepted as accurate methods, they have deficiencies due to excessively large computation time and storage requirements. The spherical harmonics method, or so-called P-N approximation, was applied by Menguc and Viskanta [12] to a multidimensional radiative heat transfer problem. Whereas the mathematical development of the P-N approximation rendered it easier to solve the radiative transfer equation, it still consumes a great deal of computation time, and a significant effort is required to rederive corresponding governing equations and boundary conditions.

The discrete ordinates method has recently received more attention on account of its efficient integration with other finite difference transport equations. This method, conceptually, belongs to a family of flux models but corrects lack of couplings among the directional intensities present in some conventional flux models. In the discrete ordinates method the radiative transfer equation is solved only in a finite number of discrete directions. The principal application of the discrete ordinates method has been in the field of neutron transport [13]. The method has subsequently been applied to numerous radiative problems [14–16] with remarkable accuracy.

Through a dimensional analysis, Sparrow and Cess [17] suggested that radiative transfer cannot be neglected in the axial direction compared with that in the radial direction for the case of 3 Pe N/16 Ar < O(1). The present study was thus directed at solving a thermally developing Poiseuille flow affected by 2-D radiation by using the discrete ordinates method. The Péclet number has been set to 200 so that the 2-D effect of radiation becomes prerequisite. After its accuracy was confirmed by a comparison with the results for the nonscattering case obtained by
the integral method [9], it has focused on analyzing the two absorbing, emitting, and anisotropic as well as isotropic scattering physical problems, which have not been previously discussed. Consequently, the effects of various radiative parameters on how to heat the flow by the hot wall, such as the conduction-to-radiation parameter, optical thickness, wall emissivity, scattering albedo, and linear anisotropic scattering coefficient, could be easily clarified to enhance our understanding of the problem.

**ANALYSIS**

The present study focuses on the interaction of radiation and convection for hydrodynamically fully developed flow in a cylindrical pipe. The governing equation for energy balance is given by

\[
\frac{\text{Pe} \text{ Ar}}{4} \frac{\partial \Theta}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Theta}{\partial r} \right) - \frac{\tau^2_d}{N} (1 - \omega_0) \left( \Theta^4 - \frac{1}{4} \int_{\Omega=4\pi} i \, d\Omega \right)
\]  

(1)

Definitions of the variables are given in the nomenclature. The fully developed laminar velocity profile, \( u = 2(1 - r^2) \), is obtained from the momentum equation.

The dimensionless form of the radiative transfer equation, which represents the balance of radiative energy passing in a specified direction through a small differential volume in an emitting, absorbing, and scattering medium, can be written as

\[
\frac{2}{\tau_d} \left( \frac{\mu}{r} \frac{\partial (\eta i)}{\partial r} - \frac{1}{r} \frac{\partial (\eta i)}{\partial \varphi} + \frac{\text{Ar}}{2} \frac{\partial i}{\partial z} \right) + i
\]

\[
= \frac{1 - \omega_0}{\pi} \Theta^4 + \frac{\omega_0}{4\pi} \int_{\Omega=4\pi} P(\Omega' \rightarrow \Omega) \, i \, d\Omega
\]  

(2)

where the linear anisotropic phase function \( P \) for scattering from the incoming direction to the outgoing direction can be given by

\[
P(\Omega' \rightarrow \Omega) = 1 + a_0(\mu \mu' + \eta \eta' + \xi \xi')
\]  

(3)

In the present study a forward \((a_0 = 1)\) as well as a backward \((a_0 = -1)\) anisotropic scattering were considered.

The temperatures at the outer wall \( \Theta_w \) and at the inlet pseudosurface \( \Theta_{in} \) are 1 and 0.6, respectively. The outlet pseudosurface temperature \( \Theta_{out} \) is calculated from the energy conservation equation, and both pseudosurfaces are assumed to be radiatively black.

In the discrete ordinates method, Eq. (2) is solved in a finite number of directions spanning a full range of the total solid angle \( 4\pi \). A discrete equation for double subscripts \( mn \), which can be obtained by the direct-differencing technique
for angular derivatives, as proposed by Carlson and Lathrop [13], can be written as

\[
\frac{2}{\tau_D} \left( \frac{\mu_{mn}}{r} \frac{\partial (r i^{mn})}{\partial r} - \frac{1}{r} \alpha_{mn+1} \frac{i^{mn+1}}{2} - \alpha_{mn-1} \frac{i^{mn-1}}{2} \right) + \frac{\text{Ar}}{2} \frac{\partial i^{mn}}{\partial z} \right) + i^{mn} 
\]

\[
= \frac{1 - \omega_0}{\pi} \Theta^4 + \frac{\omega_0}{4\pi} \sum_m \sum_n P_{m'n'} W_{m'n'} \epsilon_{m'n'} 
\]

\[m = 1, \ldots, m_M \quad n = 1, \ldots, n_m\] (4)

where the coefficients \(\alpha_{mn+1/2}\) for the angular derivative term are determined from the following recurrence formula:

\[
\alpha_{mn+1/2} - \alpha_{mn-1/2} = w_{mn} \mu_{mn} 
\] (5)

with the initial value of \(\alpha_{m1/2} = 0\).

The boundary conditions for discrete equation are represented as follows:

\[
i^{mn} = i^{m'n'} \quad \mu_{mn} = \mu_{m'n'} \quad \mu_{mn} > 0 \quad r = 0 
\]

\[
i^{mn} = \epsilon_w \frac{\Theta^4_w}{\pi} + \frac{1 - \epsilon_w}{\pi} \sum_{m'} \sum_{n'} |\mu_{m'n'}| W_{m'n'} i^{m'n'} \quad \mu_{mn} < 0 \quad r = 1 
\]

\[
i^{mn} = \epsilon_{in} \frac{\Theta^4_{in}}{\pi} + \frac{1 - \epsilon_{in}}{\pi} \sum_{m'} \sum_{n'} |\xi_{m'n'}| W_{m'n'} i^{m'n'} \quad \xi_{mn} > 0 \quad z = 0 
\]

\[
i^{mn} = \epsilon_{out} \frac{\Theta^4_{out}}{\pi} + \frac{1 - \epsilon_{out}}{\pi} \sum_{m'} \sum_{n'} |\xi_{m'n'}| W_{m'n'} i^{m'n'} \quad \xi_{mn} < 0 \quad z = \frac{2}{\text{Ar}} 
\]

where \(mn\) and \(m'n'\) denote the outgoing and incoming directions, respectively.

As long as the symmetry and invariance properties of the physical system are to be preserved, the choice of ordinate directions \(\Omega_{mn}\) and quadratic weighting factor \(w_{mn}\) is arbitrary. However, a complete symmetric quadrature is preferred because of its generality. In general, there are \(N(N + 2)\) directions for \(N = 2, 4, 6, \ldots\). This \(N\) is the accompanying letter of the commonly used S-N discrete ordinates scheme. For the present study, 24 directions have been chosen, and thus it is called the S-4 approximation. But there are only 12 independent directions, owing to the axisymmetric condition. Preliminary evaluations revealed that the S-4 approximation is quite adequate in this study, for no measurable gain in accuracy was obtained by higher order approximations such as S-6 and S-8. Thus only the S-4 approximation is used in this study. The ordinate directions and quadratic weighting factors that are used in this analysis are given in Table 1 according the work of Fivel and [15].
Table 1. Ordinate directions and weighting factors for the S-4 approximation

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>( \mu_{mn} )</th>
<th>( \eta_{mn} )</th>
<th>( \xi_{mn} )</th>
<th>( W_{mn} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-0.2959</td>
<td>0.2959</td>
<td>-0.9082</td>
<td>( \pi/3 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.2959</td>
<td>0.2959</td>
<td>-0.9082</td>
<td>( \pi/3 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-0.9082</td>
<td>0.2959</td>
<td>-0.2959</td>
<td>( \pi/3 )</td>
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<tr>
<td>2</td>
<td>2</td>
<td>-0.2959</td>
<td>0.9082</td>
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<td>( \pi/3 )</td>
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<tr>
<td>2</td>
<td>3</td>
<td>0.2959</td>
<td>0.9082</td>
<td>-0.2959</td>
<td>( \pi/3 )</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.9082</td>
<td>0.2959</td>
<td>-0.2959</td>
<td>( \pi/3 )</td>
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<tr>
<td>3</td>
<td>1</td>
<td>-0.9082</td>
<td>0.2959</td>
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<td>3</td>
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<td>4</td>
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<td>-0.2959</td>
<td>0.2959</td>
<td>0.9082</td>
<td>( \pi/3 )</td>
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<tr>
<td>4</td>
<td>2</td>
<td>0.2959</td>
<td>0.2959</td>
<td>0.9082</td>
<td>( \pi/3 )</td>
</tr>
</tbody>
</table>

By integrating Eq. (4) over the control volume as shown in Figure 1, the following equation is obtained:

\[
\mu_{mn} (A_n \xi_{mn} - A_s \xi_{sn}) - (A_n - A_s) \frac{\alpha_{mn+1/2} i_{mn+1/2} - \alpha_{mn-1/2} i_{mn-1/2}}{W_{mn}}
\]

\[
+ \frac{Ar}{2} \xi_{mn} (A_c \xi_{cn} - A_w \xi_{wn}) = \frac{\tau_D}{2} (-i_{mn} + S)V_p
\]

(7)

where the source term \( S \) is defined as

\[
S = \frac{1 - \omega_0}{\pi} \Theta^4 + \frac{\omega_0}{4\pi} \sum_{m} \sum_{n} w_{mn} i_{mn}
\]

(8)
To relate the facial intensities of the control volume and the edge intensities of the angular range to the cell center intensity, the following linear equation is used:

\[ i_p^{mn} = f i_{ic}^{mn} + (1 - f) i_{is}^{mn} = f i_p^{mn+1/2} + (1 - f) i_p^{mn-1/2} \quad \frac{1}{2} \leq f \leq 1 \tag{9} \]

In the above equation, subscript \( i \) represents the spatial direction \( r \) or \( z \). The subscripts \( s \) and \( e \) denote the starting face out of which a bundle of intensity originally comes, and the ending face at which the intensity arrives, respectively. For a positive set of \( (\mu_{mn}, \xi_{mn}) \), the subscripts \( s \) and \( e \) are assigned as \( (s, n) \) along the \( r \) direction and \( (w, e) \) along the \( z \) direction, respectively. Rearranging Eq. (7) for \( i_p^{mn} \) by substituting Eq. (9) into the intensities for the ending face and direction of \( mn + 1/2 \) results in

\[ i_p^{mn} = \frac{|\mu_{mn}| \bar{A}_r i_{rs}^{mn} + A_r/2 |\xi_{mn}| \bar{A}_z i_{zs}^{mn} - \Gamma_m \bar{\alpha}_{mn} i_p^{mn+1/2} + \tau_D/2 f V_p S}{|\mu_{mn}| \bar{A}_r + A_r/2 |\xi_{mn}| A_{ze} - \Gamma_m \bar{\alpha}_{mn+1/2} + \tau_D/2 f V_p} \tag{10} \]

where the mean coefficients are defined as

\[ \bar{\alpha}_{mn} = f \alpha_{mn-1/2} + (1 - f) \alpha_{mn+1/2} \quad \bar{\alpha}_{mn} = (\bar{A}_n - \bar{A}_s)/w_{mn} \tag{11} \]

After finding the intensity at the cell center, the intensities at the ending faces are calculated by using Eq. (9).

A numerical procedure starts by assuming that all the boundaries are black and no in-scattering term exists. With these assumptions, the factor \( f \) is gradually increased globally from its initial value of 0.5, until the negative intensity is removed. When these physically unrealistic negative intensities disappear, the real value of in-scattering effect as well as the wall emissivity are taken into account. The energy equation is discretized by finite difference approximations and solved by the tridiagonal matrix algorithm. Then global iterations are made until the solution converges. For all calculations a 21 \times 21 uniform grid system is adopted, and the convergence is checked by temperature differences between two iteration steps. A typical run requires about 4 min on a PC with a Pentium II processor to get a converged solution when underrelaxation is used.

**RESULTS**

In order to discuss the thermally developing characteristics, the mixed mean temperature and the local Nusselt number are defined as

\[ \Theta = \frac{\int \Theta u dA}{\int u dA} \quad \Theta = \frac{\int \Theta u dA}{\int u dA} \tag{12} \]

\[ \text{Nu} = \text{Nu}^C + \text{Nu}^R = -2 \frac{\partial \Theta}{\partial r} \bigg|_w + \frac{\tau_D}{N} \frac{q^R_w}{\Theta - 1} \tag{13} \]
where the wall radiative heat flux is calculated by

$$q_w^R = \sum_m \sum_n \mu_{mn} W_{mn} i^{mn}$$  \hspace{1cm} (14)

Figure 2 shows the distributions of nondimensional radiative heat flux at the wall obtained by the S-4 approximation and the integral method [9] for various values of the conduction-to-radiation parameter $N$ and optical thickness $\tau_D$ in an infinite tube with preheated zone. In general, the results are in good agreement. The minor difference results from the following fact. Practically, the numerical calculation cannot be performed in the infinite region with respect to $Z$. Therefore the calculation domain should be set in a finite region from $-Z_0$ to $Z_1$ to be heated. Echigo et al. [9] have chosen $Z_0 = 0.45D$ and $Z_1 = 1.05D$ for convection, and the radiation contribution is considered in the region from $-(Z_0 + Z_1) + Z$ to $(Z_0 + Z_1) + Z$ around an arbitrary point $Z$. Temperatures in the region outside of the domain for convection were approximated to be identical with those at $Z = -Z_0$ and $Z_1$. In the present S-4 approximation, however, we select the same domain for both convection and radiation from $Z_0 = 1.95D$ and $Z_1 = 2.55D$.

The effect of $N$ on the temperature profiles is represented in Figure 3. For $N = 0.005$ the radial temperature distribution at $z = 0.5$ is almost uniform due to the far-reaching effect of radiation from the hot tube wall. Therefore the mixed mean temperature $\overline{T}$ approaches the wall temperature faster as $N$ decreases. This behavior is qualitatively illustrated in Figure 4, in which the local Nusselt number due to conduction at the wall $\text{Nu}^C$ and its ratio with respect to the Nusselt number due to radiation $\text{Nu}^C/\text{Nu}^R$ are presented along the axial direction $z$. As shown in

![Figure 2](attachment:image.png)

Figure 2. Comparison of wall radiative heat flux along the axial direction in an infinite tube flow with preheated zone.
Figure 3. Effect of the conduction-to-radiation parameter $N$ on the radial temperature profile at $z = 0.5$ and the mixed mean temperature.

In the figure, radiation plays a more dominant role as $N$ decreases. Therefore temperature is rapidly developing by means of a salient deep-penetrating effect of radiation. In the limiting case of $N = 1$ the temperature and $\text{Nu}^C$ nearly approach those for the no-radiation case, and $\text{Nu}^C/\text{Nu}^R$ is linearly decreasing along $z$ in a log-log scale. This means that, initially, the flow is mainly heated by conduction.

Figure 4. Effect of the conduction-to-radiation parameter $N$ on the local convective Nusselt number and the Nusselt number ratio due to convection and radiation.
from the wall. Therefore, as the temperature gradient at the wall decreases downstream, \( \text{Nu}^C \) is gradually diminished along \( z \). Simultaneously, since radiation plays a more significant role, \( \text{Nu}^C / \text{Nu}^R \) decreases.

Figure 4 also shows that for \( N = 0.005 \) and \( 0.008 \), \( \text{Nu}^C \) starts to increase and reaches a plateau. These reversals result from the fact that the decreasing rate of the denominator \( \Omega - \Theta_w \) in the term \( \text{Nu}^C \) of Eq. (13) exceeds that of the numerators (wall temperature gradient). From Figures 3 and 4 the temperature contour is shown to be almost fully developed near \( z = 0.5 \) for \( N = 0.005 \) and \( 0.008 \).

The variation of total heat flux with \( N \) is illustrated in Figure 5, in which the mean Nusselt number is defined as

\[
\overline{\text{Nu}} = \int_0^1 \text{Nu}(z) \, dz
\]  

(15)

The total heat transfer remarkably decreases as \( N \) increases, since the effect of radiation is diminished. In the figure it is also shown that the total heat transfer increases as the optical diameter \( \tau_D \) increases. Thus the more radiatively active the flow medium, the larger the total heat transfer becomes. On the other hand, the total heat transfer is seen to increase as the wall emissivity \( \varepsilon_w \) increases. However, the scattering effect reduces the total heat transfer as shown in the curve of the scattering albedo \( \omega_0 \).

The effect of \( \tau_D \) on \( \Omega \), \( \text{Nu}^C \), and \( \text{Nu}^C / \text{Nu}^R \) is illustrated in Figures 6 and 7. In the figures a decrease in \( \tau_D \), without any change in \( N \), means a physical reduction of tube diameter \( D \), for the values of \( \beta \), \( \lambda \), and \( T_w \) are constant.
Although the parameter $N$ alone conceptually characterizes the relative importance of conduction in regard to radiation, it shows a different aspect when coupled with $\tau_D$. For an optically thin medium, i.e., smaller $\tau_D$, the ratio of the conductive heat flux to radiative heat flux is proportional to $N/\tau_D^2$ by ordering analysis [18]. Therefore convection becomes predominant over radiation. In the limiting case of $\tau_D = 0.01$, the temperature is developing primarily by convection,
as seen in Figures 6 and 7. For \( \tau_D = 1 \) and 2 the temperature profile is fully developed near the entrance region.

Figure 8 shows the effect of the wall emissivity \( \varepsilon_w \) on \( \text{Nu}^C \) and \( \text{Nu}^C / \text{Nu}^R \). As \( \varepsilon_w \) decreases, the intensity emitted from the wall becomes weaker, and the radiative heat flux is reduced. This makes the medium temperature rise slowly, and thus \( \text{Nu}^C \) becomes larger for smaller \( \varepsilon_w \). A slight increase in \( \text{Nu}^C \) near the exit results from the same effect of the denominator in Eq. (13) as mentioned above.

The thermal characteristics for various values of \( \omega_0 \) are represented in Figure 9. In the figure if \( \omega_0 \) is nonzero and \( a_0 = 0 \), it corresponds to isotropic scattering. Usually, scattering in the medium reduces heat absorption by the medium. Therefore as \( \omega_0 \) increases, the increasing rate of the mixed mean temperature \( \overline{\Theta} \) decreases. The temperature profile for \( \omega_0 = 0.99 \) would almost correspond to the case for convective heating only without radiation, as just a little radiant energy is transformed into thermal energy. In other words, as the scattering albedo \( \omega_0 \) decreases, \( \text{Nu}^C \) is decreasing faster along the \( z \) direction. This is because more radiant energy is transformed into thermal energy. This leads to a faster heating up of the flow medium, and thus the wall temperature gradient is decreasing faster.

In Figure 10 the radial temperature profiles at four axial positions are illustrated with a combination of such parameters as \( \tau_D \), \( \varepsilon_w \), and \( \omega_0 \). The temperature profile develops faster as \( \tau_D \) and \( \varepsilon_w \) increases or as \( \omega_0 \) decreases. This trend is also shown in Figure 6.

The effect of anisotropic scattering on the radiation Nusselt number \( \text{Nu}^R \) is depicted in Figure 11 with a variation of \( N \), \( \tau_D \), and \( \omega_0 \). Since the forward scattering intensifies the radiative heat flux, \( \text{Nu}^R \) becomes higher than isotropic scattering. On the contrary, the backward scattering diminishes the forward radia-

![Figure 8](image-url)

**Figure 8.** Effect of the wall emissivity on the local convective Nusselt number and the Nusselt number ratio due to convection and radiation.
Figure 9. Effect of the scattering albedo on the local convective Nusselt number and the Nusselt number ratio due to convection and radiation.

Figure 10. Temperature profiles at four different axial positions for a combination of the optical diameter, wall emissivity, and scattering albedo.
Figure 11. Effect of the anisotropic scattering coefficient on the radiative Nusselt number for a combination of the conduction-to-radiation parameter, optical diameter, and scattering albedo.

tive intensity, and then \( \text{Nu}^R \) becomes lower than isotropic scattering. As either \( \tau_D \) increases or \( N \) decreases, the effect of anisotropic scattering becomes more obvious due to the dominance of radiation. When \( \omega_0 \) is reduced from 0.9 to 0.5, the scattering effect is reduced, but the radiation energy, which can be transformed to thermal energy, increases. Because of this opposing behavior, the effect of \( \omega_0 \) on \( \text{Nu}^R \) is not clearly shown in the figure for \( N = 0.02 \) and \( \tau_D = 2 \).

CONCLUSION

By making use of the discrete ordinates method, the thermally developing Poiseuille flow affected by radiation has been investigated in a cylindrical tube. First, the solution accuracy of the discrete ordinates method has been verified by comparison with the integral method in an infinite tube problem with a preheated zone. It was found to be very accurate and efficient in solving the radiative transfer equation. With this advantage, the following conclusion could be drawn for thermally developing flow radiatively as well as conductively heated by a hot cylindrical wall:

1. The deep-penetrating feature of radiation makes the flow temperature profile develop faster for a smaller conduction-to-radiation parameter \( N \).
2. Total heat transfer from the hot wall to the medium notably decreases as \( N \) increases, for the role of radiation is attenuated.
3. Total heat transfer increases as the optical diameter \( \tau_D \) increases in a given range. Therefore the more radiatively active medium can extract more energy from the hot wall.
4. Whereas total heat transfer increases as the wall emissivity $\varepsilon_w$ increases, the scattering effect reduces the total heat transfer.
5. As either $\tau_D$ increases or $N$ decreases, the effect of anisotropic scattering becomes more evident.

REFERENCES